Branching Exclusion Process on a Strip

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We consider a model of stochastically interacting particles on an infinite strip of $\mathbb{Z}^2$; in this model, known as a branching exclusion process, particles jump to each empty neighboring site with rate $\lambda/4$ and also can create a new particle with rate $1/4$ at each one of these sites. The initial configuration is assumed to have a rightmost particle and we study the process as seen from the rightmost vertical line occupied. We prove that this process has exactly one invariant measure with the property that H, the number of empty sites to the left of the rightmost particle, has an exponential moment. This refines a result presented by Bramson *et al.*, who proved that for $d = 1$, H is finite with probability 1.

KEY WORDS: Microscopic interface; branching exclusion process; invariant measure; hitting time.

1. INTRODUCTION

We consider the branching exclusion process **BE**(λ) as a Markovian process ξ, with state space $\Omega = \{0, 1\}^{\mathbb{Z}^d}$. As usual, we might think of this process as particles, interacting on the sites of $\mathbb{Z}^d$, that can either jump to or create new particles at their nearest neighboring empty sites. Branching is done (in case it is possible) by existing particles after an exponential waiting time of rate 1. For any neighboring sites $x, y \in \mathbb{Z}^d$ we exchange the values at x and y at rate $\lambda/2d$; this is called a stirring process. In practice nothing happens if x and y are both empty or both occupied, but if, for instance, x is occupied and y is empty, we have that at rate $\lambda/2d$ a particle jumps from x to y. For this reason this is also called the exclusion process.

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The branching exclusion process is a Markov process defined by the generator

$$(Lf)(\xi) = \lambda \sum_{i,\, j \in \mathbb{Z}^d} p(i, j)[f(\xi^{i,j}) - f(\xi)] + \sum_{i,\, j \in \mathbb{Z}^d} p(j, i)[f(\xi^j) - f(\xi)]\, \xi(i)$$

where $f\colon \Omega \to \mathbb{R}$ is a cylindrical function, $p(j, i)$ is $1/2d$ if $\|i-j\|_1 = 1$ and 0 otherwise, and

$$\xi^{i,j}(x) = \begin{cases} \xi(i) & \text{if} \quad x = j \\ \xi(j) & \text{if} \quad x = i \\ \xi(x) & \text{otherwise} \end{cases}$$

$$\xi^j(x) = \begin{cases} 1 & \text{if} \quad x = j \\ \xi(x) & \text{otherwise} \end{cases}$$

The simplest way of defining this process is the graphical representation. The main idea is to build a structure of percolation on $\mathbb{Z}^d \times [0, \infty)$, over which we define the stirring and the growth processes. For each pair x and $y \in \mathbb{Z}^d$ such that $\|x-y\|_1 = 1$, let (i) $\{S_n^{\{x,y\}}, n \geqslant 1\}$ be Poisson processes with rate $\lambda/2d$ and at times $S_n^{\{x,y\}}$ we draw an exclusion arrow ($\leftrightarrow$) connecting x and y, and (ii) let $\{T_n^{(x,y)}, n \geqslant 1\}$ be Poisson processes with rate $1/2d$ and at times $T_n^{(x,y)}$ we draw a contact arrow ($\rightarrow$) from x to y. If at time t a double arrow appears between x and y, then the contents of x and y are exchanged. If a contact arrow appears from x to y and $\xi_t(x) = 1$, $\xi_t(y) = 0$, then a new particle is created onto y. For more details about graphical representation see, for example, Durrett[7] and Bramson *et al.*[3]

In this paper we investigate this process in an infinite strip of $\mathbb{Z}^2$. This means that we do not allow particles to leave the strip or to create new particles off the strip. We start the process with a configuration that has a rightmost particle. Bramsom *et al.*[3] proved that in $d = 1$ the branching exclusion process as seen from the rightmost particle has only one invariant measure; moreover, that this measure has the feature that the horizontal distance between X, the rightmost particle, and Z the leftmost empty site, is a random variable assuming only finite values with probability one. Cammarotta and Ferrari[6] showed that starting with this measure, the position of the rightmost particle (edge), conveniently rescaled and centered, converges to a Brownian motion. To do so, they prove that some regenerative times have a second moment finite. However, from their proof the problem of showing that under the invariant measure, for instance, the number of vacant sites to the left of the rightmost particle has first moment finite remained open. our main result fill this gap as follows. For the process in the strip we show, following Bramson *et al.*,[3] that there exists an invariant measure as seen from the rightmost particle. Furthermore, we show that the distance between the rightmost particle and the leftmost vacant site has an exponential moment. This implies that the interface region is very tight. This is probably due to the asymmetry of the motion. The same happens for the biased voter model in the strip.[11] In contrast, for the one-dimensional, not-nearest-neighbor symmetric voter model, the interface is tight in the sense that $X-Z$ is finite with probability one, but its expectation is infinite.[5] The same behavior is expected—but not proved—for the nearest neighbor voter model plus exclusion, and for the unbiased voter model in the strip.

The fact that $X-Z$ has an exponential moment implies that, conveniently rescaled, the position of the rightmost particle (edge line) converges to a Brownian motion. We also show how to use the approach presented in Bramson *et al.*[3] to obtain that the average velocity $V(\lambda)$ of the edge satisfies $\lambda^{-1/2}V(\lambda)\to 1$ as $\lambda\to\infty$, a result which holds as well for the strip with periodic boundary conditions (an infinite cylinder) and forming an angle with the coordinate axis. This is an indication that the asymptotic shape of the branching exclusion process in $\mathbb{Z}^2$ rescaled by $\sqrt{\lambda}$ is the unitary circle.[16]

2. RELATED PROCESSES

In what follows we denote by ξ_t^μ the process whose distribution at time $t=0$ is described by the measure μ; in particular we write ξ_t^η for the process starting with configuration η. We also write $\|x\|_1=\sum_{i=1}^d |x_i|$ and $\|x\|_2=(\sum_{i=1}^d x_i^2)^{1/2}$ for the norms of $x=(x_1,..., x_d)\in\mathbb{Z}^d$.

We now define three processes related to $\mathbf{BE}(\lambda)$. First of all we define

$$\zeta_t^1=\{x\in\mathbb{Z}^d: \xi_s(x)=1 \text{ for some } s\leqslant t\}$$

which is the set of sites that have already been visited by some particle of ξ_t. We also define

$$t_1(x)=\inf\{t: \xi_t(x)=1\}$$

which we call the *first hitting time* of x; then for each $x\in\mathbb{Z}^d$ we have that

$$x\in\zeta_t^1 \Leftrightarrow t_1(x)\leqslant t$$

Another process embedded in ξ_t is

$$\zeta_t^2=\{x\in\mathbb{Z}^d: \xi_s=1 \qquad \text{for all} \quad s\geqslant t\}$$

which is the set of sites that, up to time t, have been definitively occupied. The *definitive occupation time* is defined by

$$t_2(x) = \sup\{t: \xi_t(x) = 0\}$$

and, consequently, for each $x \in \mathbb{Z}^d$,

$$x \in \zeta_t^2 \Leftrightarrow t_2(x) \leqslant t$$

Observe that if $\xi_t(x) = 1$ for all t, then $t_2(x) = 0$.

Durrett and Griffeath[8] proved a general shape theorem that can be applied to a $\{0, 1\}^{\mathbb{Z}^d}$-valued Markov process having the empty set as an absorbing state and with its rates satisfying the following two conditions: translation invariance and attractiveness, which means that if $A \subset B$, then ξ_t^A and ξ_t^B can be coupled in such a way that $\xi_t^A \subset \xi_t^B$ for all t; they called it a *growth process*. With their result (see ref. 7, Chapter 11d) we obtain that $\mathbb{P}(t_2(x) < \infty) = 1$, which implies that every finite region will be occupied forever after a sufficiently large time t.

Finally we define the random walk $X_t^{\{x\}}$ starting at the site x and embedded in $\xi_t^{\{x\}}$. If we imagine that in the graphical construction, for each realization of the process we have a ramification over $\mathbb{Z}^d \times [0, \infty)$, we can tell that this random walk chooses a branch of the process $\xi_t^{\{x\}}$. That construction, applied to the asymmetric voter model, is in Bramsom and Griffeath.[4] The general idea is that $X_t^{\{x\}}$ jumps according to any exclusion arrow it encounters, but follows only the contact arrows that carry it closer to the site x. For x and t fixed, we define the following jump rates:

(i) If $X_t^{\{x\}}$ is at y, a site out of any set $R_i := \{y = (y_1,..., y_n) \in \mathbb{Z}^d: y_i = x_i\}$ for $i = 1,..., d$, then there are two possibilities for the rate of jumps from y to z such that $\|y - z\|_2 = 1$:

- If $\|z - x\|_2 < \|y - x\|_2$, then $y \to z$ at rate $(\lambda + 1)/(2d)$.
- If $\|z - x\|_2 > \|y - x\|_2$, then $y \to z$ at rate $\lambda/(2d)$.

(ii) On the other hand, when $X_t^{\{x\}}$ is at y and $y \in R_i$ for some i, then it jumps from y to z such that $\|y - z\|_2 = 1$ by one of the following two possibilities:

- If $z_i - x_i = 1$ for some i such that $x_i = y_i$, then $y \to z$ at $(\lambda + 1)/(2d)$.
- If $z_i - x_i = -1$ for some i such that $x_i = y_1$, then $y \to z$ at $\lambda/(2d)$.

Since it jumps to one half of its neighbors at rate $(\lambda + 1)/2d$ and to the other half at rate $\lambda/2d$, this process jumps at a rate $(2\lambda + 1)/2$, independent of the current position. Moreover, it has a uniformly positive drift toward the site x off of some ball D, depending on λ.

We define in the same way $X_t^{x,y}$ as a random walk starting at the site x with drift to the site y of $\mathbb{Z}^d$. Finally, we define $X_t^{x,\infty}$ as a random walk in the horizontal strip starting at the site x with drift to the right. The next result holds for both $\mathbb{Z}^d$ and the strip.

Lemma 1. For $t_2(0)=\sup\{t: \xi_t^{\{0\}}(0)=0\}$ we have

$$\mathbb{P}(t_2(0)=0)=p_2>0$$

Proof. As mentioned above, from Durrett and Griffeath[8] it follows that $\mathbb{P}(t_2(0)<\infty)=1$. Hence there exists $t<\infty$ such that $\mathbb{P}(t_2(0)<t)>0$. Then, for some fixed t,

$$\mathbb{P}(t_2(0)<t)=\sum_{\substack{A \text{ finite} \\ \{0\}\in A}} \mathbb{P}(\xi_t^{\{0\}}=A)\,\mathbb{P}(\xi_s^{\{0\}}(0)=1\ \forall s\geqslant t \mid \xi_t^{\{0\}}=A)$$

which implies that there exists a finite region A of $\mathbb{Z}^d$, enclosing the origin, such that

$$0<\mathbb{P}(\xi_s^{\{0\}}(0)=1\ \forall s\geqslant t \mid \xi_t^{\{0\}}=A)=\mathbb{P}(\xi_s^A(0)=1\ \forall s\geqslant 0)$$

where the equality comes from the Markov property. For this A we can assure that

$$\begin{aligned}
&\mathbb{P}(\xi_s^{\{0\}}(0)=1\ \forall s\geqslant 0)\\
&\quad\geqslant \mathbb{P}(\xi_t^{\{0\}}\supset A,\ \xi_s^{\{0\}}(0)=1,\ \forall 0\leqslant s\leqslant t)\ \ \mathbb{P}(\xi_s^A(0)=1\ \forall s\geqslant 0)
\end{aligned}$$

We now observe that the event $\{\xi_t^{\{0\}}\supset A,\ \xi_s(0)=1,\ \forall 0\leqslant s\leqslant t\}$ has a strictly positive probability, since it depends on a finite region of the time and space. From this we can conclude that

$$\mathbb{P}(\xi_s^{\{0\}}(0)=1\ \forall s\geqslant 0)>0 \quad \blacksquare$$

As a direct consequence of the last result, the strong Markov property, and attractiveness, we have that $\mathbb{P}(t_1(x)=t_2(x))\geqslant p_2>0$ for each $x\in\mathbb{Z}^d$.

3. EXPONENTIAL TAIL OF THE DEFINITIVE OCCUPATION TIME

The next result tells us that the definitive time of occupation of the origin is stochastically dominated by a random variable that has an exponential tail (i.e., has a distribution with a tail that decays exponentially fast) for each λ and d. The translation invariance of the dynamics implies that this upper bound works also for the distribution of the difference between the two times of occupation of any site of the lattice. The next result holds for both $\mathbb{Z}^d$ and the strip.

Theorem 2. Consider the $\mathbf{BE}(\lambda)$ starting from one particle at the origin. There exist $0 < c_1 \gamma < \infty$, depending on λ and d, for which

$$\mathbb{P}(t_2(0) > t) \leqslant Ce^{-\gamma t}$$

Consequently $\mathbb{P}(t_2(x) - t_1(x) > t) \leqslant Ce^{-\gamma t}$ for all $x \in \mathbb{Z}^d$.

Proof. For each realization of the process $\xi_t^{\{0\}}$ we define the sets of random times O_i, D_i for $i \in \mathbb{N}$ by $O_0 = D_0 = 0$ and for $i \geqslant 1$ by

$$D_i = \min\{\inf\{t > O_{i-1}: \xi_t^{\{0\}}(0) = 0\}, t_2(0)\}$$

$$O_i = \min\{\inf\{t > D_i: \xi_t^{\{0\}}(0) = 1\}, t_2(0)\}$$

We also denote $n = \max\{i: O_i - D_i > 0\}$.

Observe that n is stochastically dominated by a random variable N having geometric distribution with probability of success equal to p_2, defined in Lemma 1. A new trial of the definitive occupation, under more suitable conditions than at time $t = 0$, occurs at times O_i.

Let us take $\mathscr{E} = \{\pm e_1, ..., \pm e_d\}$, the set of the nearest neighbors of the origin. We have that for every $k \leqslant n$ there exists a random set A that has at least one of the elements of the set $\mathscr{E}$ such that $\xi_{D_k}^{\{0\}}(x) = 1$ for all $x \in A$. This happens because at the instant D_k a stirring arrow arises connecting the origin to some element y of $\mathscr{E}$ with

$$\xi_{D_k^-}^{\{0\}}(y) = 0$$

Using the strong Markov property, the attractiveness of the $\mathbf{EC}(\lambda)$, and a Feller[10] estimate to the probability of the first return to zero for a random walk with a drift toward the origin, we can guarantee that there exist $0 < C_1, \gamma_1 < \infty$ such that

$$\begin{aligned} \mathbb{P}(O_i - D_i > s) &\leqslant \mathbb{P}(\inf_t \{\xi_t^{\{e_1\}}(0) = 1\} > s) \\ &\leqslant \mathbb{P}(\inf_t \{X_t^{\{e_1\}} = 0\} > s) \leqslant C_1 e^{-\gamma_1 s} \qquad (3.1) \end{aligned}$$

and the events $\{D_1 = s\}$ and $\{O_1 - D_1 = t\}$ are related to two disconnected regions of the graphical representation, which means that we can construct an auxiliary random variable T_1, stochastically bigger than $O_1 - D_1$, with an exponential tail (3.1) and independent of D_1.

On the other hand, we can verify that at times O_i a random set B_i of sites enclosing the origin will be occupied. For $B := B_i$ (random) we define

$$t_2^B(0) = \sup\{t: \xi_t^B(0) = 0\}$$

and for attractiveness we verify that it is possible to construct a coupling with an auxiliary variable $\bar{t}_2(0)$ which has the same distribution of $t_2(0)$, is independent of D_1, and is such that $\bar{t}_2(0) > t_2^B(0)$ for any B. We may think that in the case the origin becomes vacant and then occupied again we can forget every particle outside the origin; then the time that we have to wait until the definitive time of occupation $\bar{t}_2(0)$ is bigger than it should be, $t_2^B(0)$, and independent of what happened in the past.

Using the random walk $X_t^{\{y\}}$ and the auto duality of $\mathbf{EC}(\lambda)$, and defining $\mathcal{M}(s)$ to be the set of realizations of the graphical representation such that there exists an exclusion arrow at time s joining the origin to some $y \in \mathscr{E}$, it follows that for any $y \in \mathscr{E}$ there exists $0 < C_2$, $\gamma_2 < \infty$ such that

$$\mathbb{P}(D_1 > s) = \int_s^\infty \bar{\mathbb{P}}(\{\xi_t^{\{0\}}(0) = 1, \forall t < r, \xi_r^{\{0\}}(0) = 0, \mathcal{M}(r)\})\, dr$$

$$\leqslant \int_s^\infty \mathbb{P}(\{\tilde{\xi}_t^{\{y\}}(0) = 0, \forall t \leqslant r)\, dr \leqslant C_2 e^{-\gamma_2 s} \tag{3.2}$$

The particle that was occupying the origin could jump to any $y \in \mathscr{E}$ with equal probability; for the jump to occur, that site must be vacant at time s. In order for the event $\{D_1 > s\}$ to occur, that jump must happen after time s. Here $\bar{\mathbb{P}}$ represents a probability density. Working with the dual from that y, we have that it could not have visited the origin before the dual time s. The last inequality comes when we compare the process $\tilde{\xi}_t^{\{y\}}$ to $X_t^{\{e_1\}}$ and follow Cammarotta and Ferrari (ref. 6, p. 7).

We observe now that $t_2(0)$ assumes value zero with probability p_2 (defined in Lemma 1) and assumes the value $O_1 + t_2^B(0)$ (B random and enclosing the origin) with probability $(1 - p_2)$. From that we have for $\theta > 0$ that the following inequality holds:

$$\mathbb{E}[\exp^{\{\theta t_2(0)\}}] \leqslant p_2 + \mathbb{E}[\exp^{\{\theta[D_1 + T_1 + \bar{t}_2(0)]\}}](1 - p_2)$$

and then, using the exponential Chebyshev inequality,

$$\mathbb{P}(t_2(0) > t) \leqslant e^{-\theta t} p_2 \{1 - \mathbb{E}[e^{\theta[D_1 + T_1]}](1 - p_2)\}^{-1}$$

As a consequence of the last inequality, (3.1), and (3.2) we get that, for small enough $\theta > 0$, there exist $0 < C, \lambda < \infty$ such that

$$\mathbb{P}(t_2(0) > t) \leqslant C e^{-\lambda t} \quad \blacksquare$$

4. BE(λ) WITHIN THE HORIZONTAL STRIP AS SEEN FROM THE EDGE

Bramson *et al.*[3] studied this process for $d=1$ concentrated at the set of configurations with a rightmost particle. There they proved the unicity of the invariant measure for the process as seen from the rightmost particle and the fact that it is concentrated on configurations with a finite number of vacant sites (holes) to the left of the rightmost particle. Here we show that the number of holes to the left of the rightmost particle has an exponential moment.

Let $\mathcal{N}=\{0,..., N-1\}$ and denote

$$\Omega=\{0, 1\}^{\mathbb{Z}\times\mathcal{N}}$$

and

$$\tilde{\Omega}=\{\eta\in\Omega: \eta(0, y)=1 \text{ for some } y\in\mathcal{N}, \eta(x, y)=0, \forall(x, y)\in\mathbb{N}\times\mathcal{N}\}$$

For μ on $\tilde{\Omega}$ and ξ_t^μ restricted to $\mathbb{Z}\times\mathcal{N}$ we define

$$X_t^\mu=\sup\{x\in\mathbb{Z}: \xi_t^\mu(x, y)=1 \text{ for some } y\in\mathcal{N}\}$$

and $\zeta_t^\mu(x, y)=\xi_t^\mu(x+X_t, y)$ is the translated process, as seen from the rightmost occupied vertical line; in this way, $(\zeta_t^\mu: t\geqslant 0)$ is a process with state space on $\tilde{\Omega}$ because the particles are not allowed to leave the strip or to create new particles off the strip. Next, we define for $\eta\in\tilde{\Omega}$ and $x\leqslant 0$

$$H_x(\eta)=\sum_{y\in\mathcal{N}}[1-\eta(x, y)]$$

which is the number of holes on the vertical line x in the η configuration, and

$$H(\eta)=\sum_{x\leqslant 0}[\mathbb{1}_{\{H_x(\eta)>0\}}(x)]$$

which is the number of vertical lines on the left of the rightmost vertical line occupied, with at least one hole.

Theorem 3. The process $\{(\zeta_t^\eta, t\geqslant 0), \eta\in\tilde{\Omega}\}$ has only one invariant measure ν, and there exist $0<C, \gamma<\infty$ such that

$$\nu(\{H\geqslant k\})\leqslant Ce^{-\gamma k}, \qquad \forall k\in\mathbb{N} \tag{4.1}$$

In particular, ν is concentrated on

$$\tilde{\Omega}_0 = \{\eta \in \tilde{\Omega} : \eta(x, y) = 1 \text{ if } x < M \text{ and } y \in \mathcal{N} \text{ for some } M > -\infty\} \quad (4.2)$$

Proof. The existence of the invariant measure is guaranteed by a theorem in Liggett[14] (p. 10), because the process $\{(\zeta_t^\eta, t \geq 0), \eta \in \tilde{\Omega}\}$ is Feller and $\tilde{\Omega}$ is compact (for more details see Bramson *et al.*[3] or Ferrari.[11]) The unicity is obtained as follows. Let ν be an invariant measure; any configuration with a finite number of particles is transient because particles are created and are not killed. As a consequence, ν is concentrated on configurations in $\tilde{\Omega}$ with an infinite number of particles. Defining $\tilde{t}_2(x) = \max\{t_2(x, y) : y \in \mathcal{N}\}$, it follows, as a consequence of Theorem 2, that this random variable has an exponential tail. We show below that if ν is invariant, then it must satisfy (4.1). This implies that ν must be concentrated on $\tilde{\Omega}_0$ and we finally conclude the proof because $(\zeta_t^\eta, t \geq 0)$ is a process with a countable state space, which allows us to assert the unicity of the invariant measure since under this measure the process is an irreducible Markov process on a countable state space.[10, 14]

Now we are going to put into practice our program. For $n \in \mathbb{N}$ let

$$\tilde{A}_t(n) = \{H_{-n}(\zeta_t^\eta) > 0\}$$

be the event that there is at least one vacant site at the vertical line $-n$ at time t. Moreover, let

$$\tilde{b}_t(n) = \inf\{s : H_{X_t^\nu - n}(\xi_s^\nu) < N\}$$

be the first instant that some site of the vertical line $-n$, as seen from the position of the edge at time t, was occupied. For technical reasons we set $\tilde{b}_t(n) = t$ if that never happened up to time t.

We study the displacement of the random walk which starts from some occupied position of the origin line to assure that, for t big enough, the probability of the event $\{H_{-n}(\zeta_s^\nu) < N, \text{ for some } s \leq t\}$ is very high. More precisely, for each n we make $t(n) = n^2$ in order to have that the embedded random walk starting from the origin line, with very high probability, has overtaken the vertical line n. That event assures us the occurrence of the event $\{H_{-n}(\zeta_s^\nu) < N\}$ for some $0 \leq s \leq t(n)$. Taking $x = 0$ (the origin), we construct the processes X_t^ν and $X_t^{\nu, \infty}$ together (we make a coupling) in order to have $X_t^\nu \geq X_t^{\nu, \infty}$ stochastically. As the random walk $X_t^{\nu, \infty}$ has a drift like $1/2d$ toward $+\infty$, we can ensure that for any $\varepsilon > 0$ there exist $0 < \gamma_1, C_1 < \infty$ such that

$$\nu\left(\left\{X_{t(n)}^\nu < n^2\left(\frac{1}{2d} - \varepsilon\right)\right\}\right) \leq C_1 e^{-\gamma_1 n}$$

Now for $\tilde{t}_2(x) = \max\{t_2(x, y): y \in \mathcal{N}\}$, we can verify that

$$\mathbb{P}(\tilde{t}_2(0) > t) = \mathbb{P}\left(\bigcup_{y \in \mathcal{N}} \{t_2(0, y) > t\}\right)$$
$$\leqslant \sum_{y \in \mathcal{N}} \mathbb{P}(\{t_2(0, y) > t\}) \leqslant N\mathbb{P}(\{t_2(0, z) > t\})$$

for some fixed $z \in \mathcal{N}$. As $t_2(0, z) \leqslant t_1(0, z) + t_2(0, 0)$ stochastically, where the two terms to the right have an exponential tail, we have that $\tilde{t}_2(x)$ also has an exponential tail.

Now we need the following two inequalities:

$$\nu\left(\tilde{A}_t(n) \cap \left\{\tilde{b}_t(n) \leqslant t(n) - n\left(\frac{N^{-1}}{\lambda+1} - \varepsilon\right)\right\}\right)$$
$$\leqslant \mathbb{P}\left(\left\{\tilde{t}_2(0) \geqslant n\left(\frac{N^{-1}}{\lambda+1} - \varepsilon\right)\right\}\right) \leqslant C_2 e^{-\gamma_2 n}$$

and

$$\nu\left(\tilde{A}_t(n) \cap \left\{t(n) - n\left(\frac{N^{-1}}{\lambda+1} - \varepsilon\right) < \tilde{b}_t(n) \leqslant t(n)\right\}\right)$$
$$\leqslant \mathbb{P}\left(\left\{S_n < n\left(\frac{N^{-1}}{\lambda+1} - \varepsilon\right)\right\}\right) \leqslant C_3 e^{-\gamma_3 n}$$

where $S_n = \sum_{i=1}^{n} Y_i$ is a sum of independent exponential random variables with parameter $N(\lambda+1)$. To show the first inequality in the last display, notice that, given that the line x is occupied by at least one particle, the waiting time for the line $x+1$ to be occupied dominates Y_i stochastically; the last inequality is a direct result from the large-deviations theory for random independent variables (ref. 9, p. 9). We have that

$$\nu(\tilde{A}_t(n))$$
$$= \nu\left(\tilde{A}_t(n) \cap \left\{X_t^\nu < n^2\left(\frac{1}{2d} - \varepsilon\right)\right\}\right)$$
$$+ \nu\left(\tilde{A}_t(n) \cap \left\{X_t^\nu \geqslant n^2\left(\frac{1}{2d} - \varepsilon\right)\right\} \cap \left\{\tilde{b}_t(n) \leqslant t(n) - n\left(\frac{N^{-1}}{\lambda+1} - \varepsilon\right)\right\}\right)$$
$$+ \nu\left(\tilde{A}_t(n) \cap \left\{X_t^\nu \geqslant n^2\left(\frac{1}{2d} - \varepsilon\right)\right\} \cap \left\{\tilde{b}_t(n) > t(n) - n\left(\frac{N^{-1}}{\lambda+1} - \varepsilon\right)\right\}\right)$$

and consequently

$$\nu(\tilde{A}_t(n)) \leqslant \nu\left(\left\{X_t^\nu < n^2\left(\frac{1}{2d} - \varepsilon\right)\right\}\right) + \mathbb{P}\left(\left\{\tilde{t}_2(0) \geqslant n\left(\frac{N^{-1}}{\lambda+1} - \varepsilon\right)\right\}\right)$$
$$+ \mathbb{P}\left(\left\{S_n < n\left(\frac{N^{-1}}{\lambda+1} - \varepsilon\right)\right\}\right) \leqslant Ce^{-\gamma n}$$

To conclude the proof of (4.1), we use the relation

$$\{\eta \in \tilde{\Omega} : \eta(x, y) = 1 \ \forall x < -m, \forall y \in \mathcal{N}\} \subset \{\eta \in \tilde{\Omega} : H(\eta) \leqslant m\}$$

which implies

$$\begin{aligned} &\{\eta \in \tilde{\Omega} : H(\eta) > m\} \\ &\quad \subset \{\eta \in \tilde{\Omega} : \exists (x, y) \text{ with } x < -m \text{ such that } \eta(x, y) = 0\} \\ &\quad = \bigcup_{x = m+1}^{\infty} \{\eta \in \tilde{\Omega} : H_{-x}(\eta) > 0\} \end{aligned}$$

and

$$\nu(\{H \geqslant m\}) \leqslant \sum_{i=m}^{\infty} \nu(\{H_{-i} > 0\}) \leqslant \sum_{i=m}^{\infty} Ce^{-\gamma i} = Ce^{-\gamma m}$$

finishing the proof of (4.1). As a consequence, we see that H has an exponential moment and, in particular, that $\nu(\{H = \infty\}) = 0$, which implies (4.2).

5. THE ASYMPTOTIC VELOCITY OF THE EDGE

One can define the microscopic velocity of the branching exclusion process on a strip in many equivalent ways; this was pointed out by Kerstein,[12] who also made many interesting computations. With a rigorous approach, Bramson *et al.*[3] proved that for $d = 1$ (here $|\mathcal{N}| = 1$)

$$\lim_{\lambda \to \infty} \frac{\mathbb{E}(X_t^\nu)}{t\sqrt{\lambda}} = \sqrt{2}$$

where ν is the invariant measure of the system as seen from the edge. Their proof relies on a mean-field limit theorem stated in De Masi *et al.*[17] and on a large-deviation result for the branching exclusion process. By adapting their ideas when we consider $\tilde{X}_t^\nu$ to be the horizontal coordinate of the position of the rightmost occupied site of $\tilde{\mathcal{V}}_N$, the connected strip (an infinite cylinder), we can prove that, for every N,

$$\lim_{\lambda \to \infty} \frac{\mathbb{E}(\tilde{X}_t^\nu)}{t\sqrt{\lambda}} = 1 \tag{5.1}$$

where ν is the invariant measure of the system, as seen from its rightmost occupied line (the existence of ν is guaranteed as a consequence of Theorem 3 and the change of the constant is a matter of scale). Moreover, we can rotate the strip and define $\tilde{Y}_t$ to be the number of particles created inside the strip $\tilde{\mathcal{V}}_N$ rotated by $\arctan(p/q)$ (where $0 < q \leqslant p$ and $p, q \in \mathbb{N}$) up to time t and assuming the one-sided initial configuration; essentially by Theorem 3 we can assure that

$$\mathbb{E}_\nu(\tilde{X}_t) = \frac{\sqrt{p^2 q^2}}{N\sqrt{p+q}}\, \mathbb{E}_\nu(\tilde{T}_t)$$

and then follow Bramson *et al.*[3] to show that (5.1) holds for the rotated strip.

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